PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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A Survey of the Origins and Physical Importance of Soliton Equations

J. D. Gibbon

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A survey of the origins and physical importance of soliton equations

By J. D. Gibbon
Department of Mathematics, Imperial College, London SW7 2BZ, U.K.

An introduction to the subject is given in an elementary way for the non-specialist, outlining why many completely integrable systems, although special, play a significant role in wave motions in applied mathematics and theoretical physics.

1. Introduction

On a recent visit to Japan I had a chance to view a graph that showed the fluctuations in the number of papers published concerning solitons. Over the last seventeen years, the number has amounted to many thousands and still shows little sign of abating although the 'soliton gradient', while still positive, is less steep than it was five years ago.

In the earlier days of the subject in the late 1960s and early and middle 1970s much interest was aroused in parts of the scientific community by the mildly romantic story of John Scott Russell's discovery of the soliton, which was first published in Scott Russell (1844). Many of the early papers were motivated by problems arising in plasma physics, nonlinear chains or fluid dynamics. It has become increasingly clear, however, that the subject has moved on from there and a great wealth of literature of a purer mathematical nature related to Lie algebras, and particularly Kac–Moody algebras, has grown up in an attempt to address some of the fundamental problems that need to be answered concerning the nature of integrability in partial and ordinary differential equations. This Discussion Meeting is an attempt to identify some of the important areas of new development on both the pure and applied sides of the subject. Anyone present at any of the big meetings over the last decade, in which algebraic geometers and experimentalists have rubbed shoulders, will attest to the enormous diversity of the subject! The application of completely integrable systems (on which I will enlarge later) has spread rapidly into fundamental particle physics, plasma and fluid dynamics, statistical mechanics, many areas of solid state physics, biology, laser and fibre optics. The association of completely integrable systems with Lie groups and algebras and the rich variety of problems that occur on the periodic domain where concepts in algebraic geometry are necessary, opens up a totally different area of the subject to the pure mathematician. The enormous spread of the subject can often cause problems for the new or fringe participant at a meeting at which a broad selection of these wares are on view. Clearly I cannot hope to even mention all these topics, yet alone give them a cursory introduction. The papers by Dr Ward and Sir Michael Atiyah in this symposium discuss multidimensional problems, including the self-dual Yang–Mills and Bogomolny equations, and problems of this type I will leave to them. The paper by Dr Ercolani and Professor Flaschka deals with some of the algebro-geometric ideas necessary in studying the periodic problem and again I will leave this topic to them.

My main concern will be to show how many of the main partial differential equations (p.d.es), which have the genuine soliton property, fit into a framework in applied mathematics and theoretical physics that is very broad in application. It is wrong to suppose that because the number of equations that have this property is small, the subject is no more than a curiosity. As I will attempt to explain later, various scaling methods produce certain equations over again in whole ranges of subjects precisely because the same type of dispersive wave phenomena seem to occur almost universally. In many cases, it is precisely these equations that are solvable by what is now known as the inverse scattering transform (i.s.t). It is on this idea that I will concentrate in the first half of this paper, along with a brief explanation of the i.s.t. The paper is written for the reader new to the subject, not for the specialist. Because of the diversity of the applications I have necessarily had to be selective in my topics.

Our main concern will be to study how certain classes of p.d.es and differential difference equations arise and how they can be solved. We will restrict ourselves to Hamiltonian systems and so those with which we will deal must be strictly dissipationless.

In a general sense therefore, we will be dealing with infinite-dimensional Hamiltonian systems from which one would not normally expect more than three integrals of the motion: mass, momentum and energy. In some finite-dimensional Hamiltonian systems, only one integral may exist. For fluid or plasma problems, the basic equations of motion are usually of the continuity type and these integrals are easily found but solutions of the equations as they stand are often impossible when dispersion is included. Except in very unusual circumstances (Benney 1973; Gibbons 1981) the ideas we will be discussing are not applicable to nonlinear, dispersionless p.d.es, which usually give breaking solutions.

Although most full systems of nonlinear dispersive p.d.es have only a restricted number of symmetries (and hence conserved quantities) they can be reduced, by certain types of perturbation theories, to p.d.es that have an infinite number of integrals. If these integrals are in involution (i.e. they commute) then the reduced system is said to be 'completely integrable'. The main example, which is elaborated in §3, comes from trying to find which equation, or equations, governs the motion of small amplitude waves in dispersive nonlinear media. Over the years, methods have been devised for reducing these often large and intractable sets of equations to something that can be handled more easily, even numerically. Such weakly nonlinear perturbation methods generally come under the blanket title of 'reductive' perturbation theory, which will be explained in §3. From a whole range of problems attempted independently in the 1960s in plasma physics and fluid dynamics, the evidence grew that the commonest equation governing small-amplitude long-wave phenomena was either

$$u_{xxx}+\alpha u u_x+u_t=0 \tag{1.1}$$

or

$$v_{xxx}+\beta v^2 v_x+v_t=0. \tag{1.2}$$

Equation (1.1) has been known since 1895 when it was derived by Korteweg & de Vries (1895) when they studied the evolution of long waves on shallow water, and consequently (1.1) has become known as the K.d.V. equation. Equation (1.2), named with less imagination, has become known as the modified K.d.V. equation or m.K.d.V. equation. Miura (1968), discovered that there exists a transformation between the two equations

$$\alpha u=\pm(6\beta)^{\frac{1}{2}} v_x-\beta v^2, \tag{1.3}$$

which takes the form of a Riccati equation and is therefore linearizable. Furthermore since (1.1) is invariant under the Galilean transformation, $x \to x - \alpha\lambda t$; $t \to t$; $u \to u - \lambda$ (λ constant), we observe that the transformation $v = \mp (6/\beta)^{\frac{1}{2}} \psi_x/\psi$ reduces (1.3) to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\alpha}{6}u\right)\psi = \tfrac{1}{6}\alpha\lambda\psi \tag{1.4}$$

in the new coordinate frame. Since α can be scaled in (1.1), a convenient choice of $\alpha = -6$ turns (1.4) into the Schrödinger equation of quantum mechanics

$$\left(-\frac{\partial^2}{\partial x^2} + u\right)\psi = \lambda\psi, \tag{1.5}$$

where the time dependence of ψ can be found from (1.2) and comes out to be

$$\frac{\partial\psi}{\partial t} + \frac{\partial^3\psi}{\partial x^3} - 3(u+\lambda)\frac{\partial\psi}{\partial x} = f\psi, \quad f \text{ constant.} \tag{1.6}$$

This result, found by Gardner *et al.* (1967), opened the way to a whole new study of equations. We can immediately see from (1.5) that u acts as a potential in a scattering problem which has a *constant energy spectrum* no matter how u changes with t. The results of scattering theory are now available to us, particularly the work of Gel'fand & Levitan (1955) in scattering theory which shows how to reconstruct $u(x, t)$ from given scattering data (i.e. the discrete spectrum and the reflection and transmission coefficients). Gardner *et al.* (1967) mapped out how this could be done to find $u(x, t)$ from initial data. Around the same period the same group showed that the K.d.V. and m.K.d.V. equations had an infinity of integrals of the motion and subsequently Zakharov & Faddeev (1972) showed that these were in involution and so the system was formally 'completely integrable' in the Hamiltonian sense.

These results, fascinating as they are, would not have had the impact they have if they were confined to just the K.d.V. and m.K.d.V. equations. The Miura transformation, at first sight, looks like one of those accidents in science that has no application elsewhere: in fact this is not so. Lax (1968) recast the problem in a different way and in doing so opened up the idea of associating a spectral problem with a constant spectrum to other equations.

Let L be a differential operator with spectrum λ so that

$$\mathrm{L}\psi = \lambda\psi. \tag{1.7}$$

We require the evolution of L, which can be deformed in time t, such that its spectrum λ remains constant. We call this an isospectral deformation. Differentiation of (1.7) with respect to t gives

$$\mathrm{L}_t\psi + \mathrm{L}\psi_t = \lambda\psi_t \tag{1.8}$$

and we easily see that if the time evolution of ψ is described by the introduction of another operator P such that

$$\psi_t = \mathrm{P}\psi \tag{1.9a}$$

then we have

$$\mathrm{L}_t = \mathrm{PL} - \mathrm{LP} = [\mathrm{P}, \mathrm{L}]. \tag{1.9b}$$

Let L be the operator

$$\mathrm{L} = -\partial^2/\partial x^2 + u(x, t), \tag{1.10}$$

which is symmetric, and let us take the most trivial antisymmetric operator for P, namely $\mathrm{P} = -\partial/\partial x$. We obtain no more than $u_t + u_x = 0$, which is the trivial unidirectional wave

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

equation. The next suitable antisymmetric form for P, in which we have adjusted the coefficients for simplicity, is

$$P = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x + f(t). \tag{1.11}$$

Equations (1.6) and (1.11) are the same if λ is substituted from (1.5). Introduction of (1.10) and (1.11) into the Lax commutation relation gives the K.d.V. equation. For boundary conditions $u(x,t) \to 0$ as $|x| \to \infty$, the single solitary wave solution can be found easily from integrating the K.d.V. equation directly:

$$u = -\tfrac{1}{2}a^2 \operatorname{sech}^2 \tfrac{1}{2}(ax - a^3 t + \delta), \tag{1.12}$$

where a and δ are arbitrary. This specializes the 'cn^2' Jacobian elliptic solution for free end boundary conditions. Another expression in a stationary frame is the Weierstrassian elliptic function $u = -2\mathscr{P}(x)$. The origin of these elliptic function solutions and their generalizations is difficult to explain simply. The origin of the soliton solutions is easier. The spectral problem (1.5), in which λ is the 'energy' eigenvalue, will have a bound (negative) energy spectrum, which is discrete, and a continuous (positive) energy spectrum ($\lambda = k^2$), both of which are determined from the initial data $u(x,0)$. The amplitude function a_n of each soliton corresponds directly to a discrete eigenvalue ($a_n = 2\kappa_n$). Consequently, if some given initial data has no discrete eigenvalues, then no solitons will emerge. The continuous spectrum corresponds to dispersive oscillatory waves that have a self-similar structure. The word 'soliton' was indeed coined by Zabusky & Kruskal (1965) as a solitary wave that behaved in a particle-like fashion. Figure 1 shows a numerical solution displaying the interaction of two K.d.V. solitons, set well apart initially. The preservation of the wave after collision is due to the constancy of the spectrum.

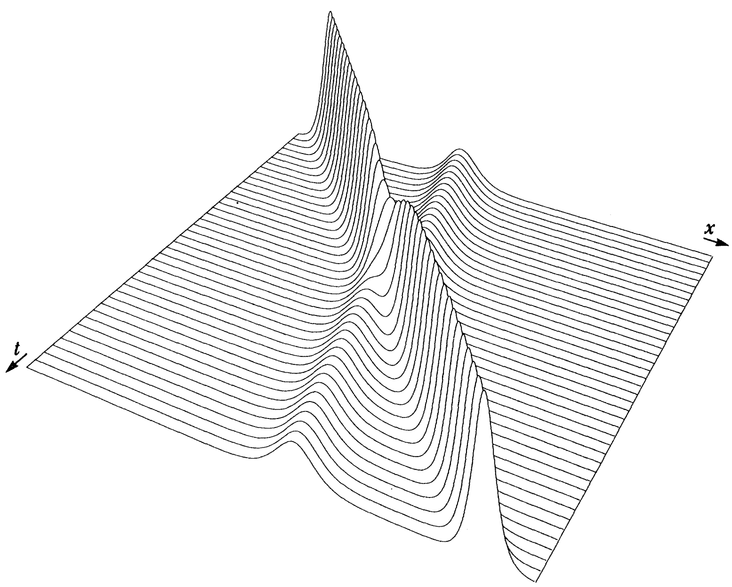

FIGURE 1. Collision of two K.d.V. solitons. Initially the larger wave starts on the left and then overtakes the smaller after collision in the centre of the picture.

It is at this point that we can run into the different usage of the word 'soliton'. The solid state physicist or field theorist might use the word to describe a solitary wave that takes the system from one vacuum state to another in finite energy as in the φ^4 equation

$$\varphi_{xx} - \varphi_{tt} = 2\varphi^3 - \varphi. \tag{1.13}$$

This equation has vacuum states at $\varphi = 0$, $\varphi = \pm 1/\surd 2$, and the solitary wave (known as a kink)

$$\varphi = \pm(1/\surd 2)\tanh[\gamma(x - vt)/\surd 2], \quad \gamma^2 = (1 - v^2)^{-1} \tag{1.14}$$

does indeed take the system from $\varphi = -1/\surd 2$ to $\varphi = +1/\surd 2$ in finite energy. The sine–Gordon equation

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi \tag{1.15}$$

with an infinity of stable vacuum states $\varphi = 0 \pmod{2\pi}$ also has a solitary wave solution

$$\varphi = 4 \arctan \exp[\gamma(x - vt)], \tag{1.16}$$

which does the same thing. However, the φ^4 equation shows no evidence in figure 2*a*, *b*, numerical or otherwise, of displaying truly elastic particle-like behaviour. The kinks (1.14) lose energy on collision and decay in amplitude, i.e. the collision is not elastic. In contrast, the sine–Gordon equation is well known to possess the same properties as the K.d.V. equation, i.e. it is associated with an isospectral problem (different from (1.5)). The use of the word soliton in both contexts therefore seems confusing as one equation is completely integrable and the

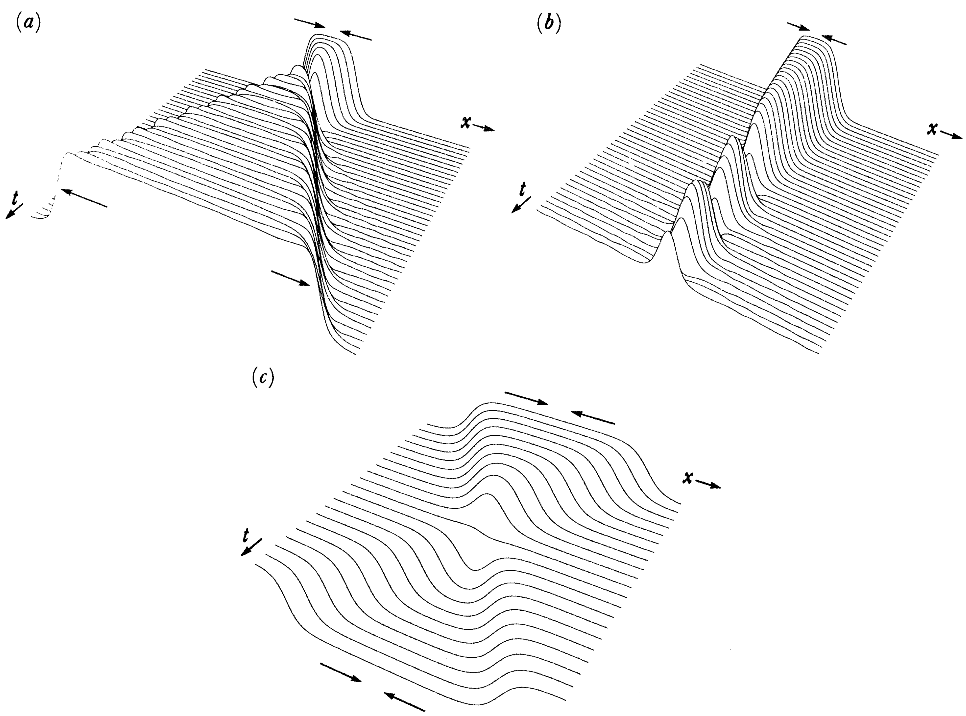

FIGURE 2. (*a*) High-energy collision of a ϕ^4 kink and anti-kink; (*b*) low-energy collision of a ϕ^4 kink and anti-kink; (*c*) head-on collision of two sine–Gordon kinks. Note the elastic behaviour in comparison with the ϕ^4 equation.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

other is not. The concept of the soliton is nevertheless valuable even in non-integrable cases because it describes finite energy motions between stable equilibrium states *that cannot be reached by perturbation theory*. The linear view of physics generally considers excitations about a well defined stable ground state; for example, phonons in a lattice. Use of perturbation theory will identify the normal modes of the system about this ground state. The soliton idea is valuable because it enables us to consider stable states of a *nonlinear* system which are both localized, which have finite energy and which maintain their identity under the influence of other excitations. While the word 'soliton' has become universal, the idea of the soliton has different meanings and names in different parts of physics. Solitary waves, kinks, monopoles, instantons, fluxons, vortices, etc. are just some of the names that describe finite-energy localized solutions of nonlinear field equations in various corners of physics. To the physicist who is interested in these types of nonlinear excitations, the exact integrability of the system is not important because the very existence of a stable, local, finite, nonlinear energy excitation (a soliton) can drastically change the partition function of a system. For a discussion of this type of problem the reader is recommended to read the review article by Bishop *et al.* (1980) and the articles in Bishop & Schneider (1978). In the sense I have defined above, in which we may be considering systems with maybe 10^{23} degrees of freedom the aim of the theoretical physicist studying condensed matter or field theory (say) may be entirely different from the applied or pure mathematician. The applied mathematician is generally interested in solving the initial value problem for a p.d.e. wherever possible. The integrability of the equation is therefore important and consequently the classification of integrable p.d.es and o.d.es of applied mathematics is of interest. The pure mathematician is also interested in integrable systems, but more from the sense of what integrability actually *means* as opposed to the properties of specific equations.

To ask if any given p.d.e. is completely integrable is a question to which there is not yet a full answer. There are two tests that give some idea whether a p.d.e. can be solved in this fashion, but neither is definitive and I will defer remarks on these until later. There are, however, certain classes of equations that are well understood and which also, importantly, have direct physical significance. I have already indicated how the K.d.V. and m.K.d.V. equations are the 'canonical' equations for the weak evolution of long waves (for example, like solitary waves). This type of wave motion is important because it describes how compressive pulses evolve in plasmas, water waves and elastic rods and strings, for example. Another even more common type of wave motion is oscillatory in nature. Various perturbation procedures, such as the method of multiple scales, show how to find the evolution of the slowly varying envelope $A(\overline{X}, T_2)$ of a packet of oscillations. In closed nonlinear dispersive systems, the nonlinear Schrödinger (n.l.S.) equation

$$\left.\begin{aligned} 2\mathrm{i}\frac{\partial A}{\partial T_2}+\left(\frac{\partial^2\omega}{\partial k^2}\right)\frac{\partial^2 A}{\partial \overline{X}^2}+\beta A\,|A|^2 = 0, \\ \overline{X} = \epsilon(x-c_g t), \quad T_2 = \epsilon^2 t, \quad \omega=\omega(k) \end{aligned}\right\} \tag{1.17}$$

is generally the evolution equation that governs the modulation of an almost monochromatic wave. Benney & Newell (1967) and Newell (1974) have shown how a very wide class of systems will yield (1.17) for small amplitude wave packets. It turns out that the n.l.S. equation is also completely integrable like the K.d.V. equation. Zakharov & Shabat (1972) showed that a matrix formulation of the Lax pair principle is required. Solitons are possible (not possible)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

when $\beta\, \partial^2\omega/\partial k^2 > 0$ (< 0) for boundary conditions $A \to 0$ as $|\bar{X}| \to \infty$. Dr Mollenauer's paper (this symposium) on the soliton laser uses this equation. An optical fibre is a dispersive, almost lossless medium down which light in very short pulses (picosecond duration or less) can travel. For certain materials and frequencies, $\beta\, \partial^2\omega/\partial k^2 > 0$ and soliton formation occurs in the fibre. This is then fed through an amplifier and the pulse is reshaped again before being re-cycled again through the fibre loop. The combination of the two processes produces considerable pulse narrowing. The optical fibre is an exciting application of n.l.S. solitons, which almost certainly will become very important in fibre optical devices. Nonlinear optics is a fertile ground for interesting nonlinear phenomena. It even turns out that the lossless optical amplifier is an integrable system. Certain problems in optics such as the attenuation or amplification of ultra-short pulses (modulating carrier waves usually in the visual frequency range) give rise to the sine–Gordon equation mentioned previously in (1.15) (Lamb 1971). In characteristic coordinates this is (+ for amplifier; − for attenuator)

$$\left.\begin{aligned} \varphi_{\xi\tau} = \pm \sin\varphi, \quad \varphi \to 0 \quad (\mathrm{mod}\ 2\pi), \\ |\xi| \to \infty. \end{aligned}\right\} \tag{1.18}$$

The sine–Gordon equation, as well as its role as a model field equation in classical field theory and as a pinning model in solid state physics, is also a sister equation to the n.l.S. equation. It occurs in the weakly nonlinear long-time and space-scale limit of those dispersive systems that are coupled to external sources of potential energy (Gibbon *et al.* 1979; Gibbon & McGuinness 1981) (the n.l.S. equation describes closed systems in equilibrium).

These are three quite general types of wave motion, the first two of which at least are very broad indeed in their application. Another very common phenomenon is that of wave resonances. These usually take three forms.

(i) *Three-wave mixing*. This involves the coupling of 3 frequencies and wavenumbers in a triad. This type of phenomenon is very common in physics and can be found in light scattering in crystals, laser cavity devices, electric circuits, water waves and plasma problems. The resulting equations for the 3-wave envelopes of this quadratic resonance phenomenon are solvable by inverse scattering (Zakharov & Manakov 1976; Kaup 1976; see also Craik 1978, 1984).

(ii) *Second harmonic resonance*. This resonance occurs when two wavenumbers k_1 and k_2 satisfy $2k_1 = k_2$ and $2\omega_1 = \omega_2$ and can occur in any wave dispersion problem if the dispersion relation has the right shape. The wave envelope equations for this problem are partly solvable by inverse scattering (Kaup 1978).

(iii) *Long-wave–short-wave resonance*. If instead of considering the evolution of long waves and wave packets as separate problems, we look for spatial and temporal scales that will allow a coupling, it is found to be possible in some problems if the group velocity of the wave packet (short wave) equals the phase velocity of the long wave. The inverse scattering problem is difficult since the Lax pair is a 3×3 problem (Yajima & Oikawa 1976).

This latter resonance brings us to a more general problem in the coupling of long and short waves. Zakharov (1972) proposed a model that coupled electron density oscillations (Langmuir waves) to ion sound waves. The resulting equations (in one dimension) are not integrable by i.s.t. as they stand. In fact very little is known about them, except that in the limit when the ion sound speed approaches infinity, they reduce to the n.l.S. equation, which *is* integrable. Rather curiously, Zakharov's equations arise in Dr Scott's paper (this symposium) on the propagation of solitons in polypeptides, in which the elastic sound waves in the helical protein structure are coupled to the dipole–dipole interactions arising from a resonance in a particular C=O bond in the molecule. In §3 I give reasons why these equations arise in this problem.

In the above paragraphs I have mentioned five or six major types of wave propagation and coupling, which spread across a wide variety of problems. The bulk of these yield equations solvable by the i.s.t., usually in some weakly nonlinear limit. This is the remarkable fact in this subject, which cannot be ignored and which shows how wide is the application of soliton theory. I have concentrated mainly on the 'classical' applications of solitons. The integrability of the self-dual Yang–Mills equations (even though the manifestation of the solutions is different) and certain equations in relativity, solid state physics and even quantum statistical mechanics attest even more to their importance. In classical physics and mathematics, most of the original equations are not integrable. By various forms of perturbation theory integrable equations are produced. These have an infinity of integrals where the underlying full system probably has only three: mass, momentum and energy. Why these scaling methods introduce extra symmetries that were not there in the first place is not fully understood and is a problem that needs to be answered.

Our main concentration has been until now, on classes of dispersive wave problems giving rise to p.d.es. It is obvious that the idea of the Lax pair will extent to ordinary matrices with time-dependent elements. In this fashion Moser (1975) proved the integrability of the system of unit masses on the line, each under an inverse square potential from all the others. The matrix L is now an isospectral matrix instead of a differential operator. From this the integrals of the system can be calculated. The idea of isospectral matrices was first used by Flaschka (1974) and Manakov (1975), who independently proved the integrability of the Toda lattice in this way. For the applied mathematician who is interested in solving problems, the idea of the Lax pair is fundamental. Given a set of equations that arise in some problem in applied mathematics, the question arises, how do we know whether a Lax pair exists or not? There is no absolutely definitive answer to this question (yet). Two methods exist that go some way towards an answer. The first is a direct method of calculating the Lax pair from the equation (if it exists), which was first designed by Wahlquist & Estabrook (1975). The method yields a Lie algebra (if the system is integrable), the representation of which will give the Lax pair. I will illustrate this idea in §4 with an example. The pure mathematician would then prefer to turn the problem around on its head and consider the classification of Lie algebras as fundamental to the understanding of which equations are integrable. The papers by Dr Wilson and Professor Frenkel to a certain degree are concerned with this fundamental approach. This area has blossomed over the last few years and to the applied mathematician seems as technical as some of his methods do to the pure mathematician. Nevertheless, it is clear that Lie algebraic ideas, particularly Kac–Moody algebras, are fundamental to a deep understanding of integrability in nonlinear problems.

Finally, there has been some considerable interest in the idea of the so-called 'Painlevé property' as a test for integrability, particularly since Ablowitz & Segur (1981) showed how the Painlevé transcendents were connected with the K.d.V. and other equations through their similarity variables. As a concept, the idea can be used to generalize the idea of integrability to both p.d.es as well as o.d.es. These ideas go back to Kovalevskaya's (1890) integration of a rigid body (see Professor van Moerbeke's paper (this symposium)). I will spend a little time in §5 discussing these since they provide a picture that unifies some of these ideas with methods for integrating finite dimensional o.d.es such as the Hénon–Heiles equations.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2. Inverse scattering for K.d.V., Hamiltonian formulation and other integrable equations

(a) *Inverse scattering*

The elegant result that the potential $u(x,t)$ of a Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2}+u(x,t)\right]\psi = \lambda\psi \tag{2.1a}$$

must evolve according to the K.d.V. equation if ψ satisfies

$$0 = \frac{\partial\psi}{\partial t}+4\frac{\partial^3\psi}{\partial x^3}-6u\frac{\partial\psi}{\partial x}-3u_x\psi-f\psi \tag{2.1b}$$

is of little use unless one can solve for $u(x,t)$. Normally in quantum mechanics we are given u and require ψ subject to the condition that ψ is square integrable. Here we have boundary and initial conditions on $u(x,t)$ (the first of which will determine ψ asymptotically) and we require $u(x,t)$. This requires us to solve what are known as the direct and inverse scattering problems. The procedure is a little messy so I will only give the bare outlines and relegate even some of that to an Appendix. First we must find the scattering data, which we do as follows. Let us consider boundary conditions $u(x,t)\rightarrow 0$ as $|x|\rightarrow\infty$, with some sufficiently smooth initial data $u(x,0)$. Then we may define eigenfunctions (Jost functions)

$$\left.\begin{aligned}\Phi &\sim e^{-ikx}, \quad x\rightarrow-\infty,\\ \left.\begin{aligned}\Psi &\sim e^{ikx},\\ \overline{\Psi} &\sim e^{-ikx},\end{aligned}\right\}& \quad x\rightarrow+\infty,\end{aligned}\right\} \tag{2.2}$$

where the energy eigenvalue $\lambda = k^2$. Since the Wronskian $W(\Psi,\overline{\Psi}) = 2ik$ and is non-zero for $k\neq 0$, Ψ and $\overline{\Psi}$ are linearly independent and we may write

$$\Phi = a(k,t)\,\overline{\Psi}+b(k,t)\,\Psi. \tag{2.3}$$

Equation (2.3) is the scattering theorists' way, in k-space, of matching a wavefunction $\overline{\Psi}$ travelling to the left and scattering into Φ and back-scattering into Ψ. The quotients $1/a$ and b/a are known as the transmission and reflection coefficients. We need to determine these and the energy spectrum in terms of the initial conditions. Initial data $u(x,0)$ will determine λ and since λ is constant it is determined forever. From (2.1a) we have a continuous positive energy spectrum $\lambda = k^2$ and a negative bound energy spectrum $\lambda = -\kappa_n^2$, which is discrete. We can now determine a and b. For instance, using the continuous spectrum we have

$$\left.\begin{aligned}\psi &\sim e^{-ikx}, & x&\rightarrow-\infty,\\ \psi &\sim a\,e^{-ikx}+b\,e^{ikx}, & x&\rightarrow+\infty.\end{aligned}\right\} \tag{2.4}$$

In (2.1b) we now easily find that

$$a_t = 0; \quad b_t = (8ik^3)\,b \tag{2.5}$$

and that $f(t) = 4ik^3$. Consequently we now have $a(k,t)$ and $b(k,t)$ *in terms of their initial values*

$$a(k,t) = a(k,0); \quad b(k,t) = b(k,0)\exp(8ik^3t). \tag{2.6}$$

Hence, given initial data $u(x,0)$ we can, at least in principle, determine the bound states κ_n, the reflection and transmission coefficients $b(k,0)/a(k,0)$ and $a^{-1}(k,0)$, and we already know the dispersion relation $\omega = -k^3$. These are the scattering data and are sufficient to determine the evolution *for all t*.

The next step, the inverse problem, is more tricky and much of the calculation must be relegated to Appendix I. Suffice to say that the eigenfunctions Ψ and $\overline{\Psi}$ may be written in the integral representations

$$\Psi = \exp(\mathrm{i}kx) + \int_x^\infty K(x,y)\exp(\mathrm{i}ky)\,\mathrm{d}y, \tag{2.7a}$$

$$\overline{\Psi} = \exp(-\mathrm{i}kx) + \int_x^\infty \overline{K}(x,y)\exp(-\mathrm{i}ky)\,\mathrm{d}y, \tag{2.7b}$$

where, notably, K and $\overline{K}$ are k-independent. The kernels K and $\overline{K}$ turn out to be equal and satisfy the linear integral equation

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z)\,B(z+y)\,\mathrm{d}z = 0, \tag{2.8a}$$

where
$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(k,0)}{a(k,0)}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k + \sum_{n=1}^{N} c_n\,\mathrm{e}^{-\kappa_n x}, \tag{2.8b}$$

where $a(k,0)$ and $b(k,0)$ have been defined above and $c_n = c_n(0)\exp(8\kappa_n^3 t)$. Equation (2.14) is known as the Gel'fand–Levitan equation since it was derived in scattering theory by them (Gel'fand & Levitan 1955; Marchenko 1952*a*, *b*). The potential $u(x,t)$ is reconstructed from $K(x,y)$ by

$$u(x,t) = -2\,\mathrm{d}K(x,x)/\mathrm{d}x. \tag{2.9}$$

This procedure is not as painful as it looks! For instance, we see that the continuous and discrete parts of the spectrum in $B(x)$ are separate. Let us take the reflection coefficient $b = 0$ and choose

$$K(x,y) = \sum_{n=1}^{N} F_n(x,t)\,\mathrm{e}^{\kappa_n y}, \tag{2.10}$$

then (2.8) reduces to a set of N linear equations in the F_n:

$$0 = F_n + c_n(t)\,\mathrm{e}^{\kappa_n x} + \sum_{m=1}^{N} c_m(t)\,\frac{\exp[(\kappa_n+\kappa_m)\,x]}{(\kappa_n+\kappa_m)}\,F_m. \tag{2.11}$$

Solving for these, (2.10) and (2.11) give, after some algebra,

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\det M, \tag{2.12a}$$

$$M_{ij} = \delta_{ij} + \frac{2(a_i a_j)^{\frac{1}{2}}}{a_i + a_j}\exp[(\theta_i+\theta_j)/2], \tag{2.12b}$$

$$\theta_i = a_i x - a_i^3 t + \delta_i, \quad a_i = 2\kappa_i. \tag{2.12c}$$

This formula indeed gives the whole set of soliton solutions of K.d.V. including the single solitary wave solution. Rational solutions of the K.d.V. equation can be found by taking certain limits in the N-soliton formula (2.12) but these solutions are best expressed in the form given by Adler & Moser (1978):

$$u = -2\frac{\partial^2}{\partial x^2}\ln \Theta_n \tag{2.13a}$$

and the Θ_n are found by recursively solving the Wronskian relation $W(\Theta_{n+1}, \Theta_{n-1}) = (2n+1)\,\Theta_n^2$, where $\Theta_0 = 1$ and $\Theta_1 = x$. At each stage we can introduce an integration factor which must be t-dependent. The next two Θs are

$$\Theta_2 = x^3 + 12t, \tag{2.13b}$$

$$\Theta_3 = x^6 + 60x^3t - 720t^2. \tag{2.13c}$$

How the various elliptic function solutions are found is an entirely different problem as the above inverse scattering procedure depended upon the boundary conditions $u \to 0$ as $|x| \to \infty$. The papers by Novikov (1974) and Lax (1975, 1976) deal with the periodic problem for the K.d.V. equation, although the paper in this symposium by Dr Ercolani and Professor Flaschka contains a more up-to-date set of references.

(b) *Hamiltonian formulation and conservation laws*

To put the K.d.V. equation in Hamiltonian form we require a class of equations of the type (see Gardner 1971 and Gardner *et al.* 1974)

$$u_t = -\frac{\partial}{\partial x}\left(\frac{\delta H_K}{\delta u}\right), \quad Q_K(u) = \frac{\delta H_k}{\delta u}. \tag{2.14}$$

Then there exists a recursion relation between the Q_K,

$$\frac{\partial}{\partial x} Q_{K+1}(u) = \left(\frac{\partial^3}{\partial x^3} - 2u\frac{\partial}{\partial x} - 2\frac{\partial}{\partial x}u\right) Q_K(u), \tag{2.15}$$

with a sequence of conserved quantities

$$H_K = \int P_K(u, u_x, \ldots)\, \mathrm{d}x. \tag{2.16}$$

If we begin with $Q_1 = u$ (i.e. with the equation $u_t + u_x = 0$) then we easily find that

$$Q_2 = u_{xx} - 3u^2, \tag{2.17a}$$

$$Q_3 = u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3. \tag{2.17b}$$

Consequently, the K.d.V. equation is the second in the hierarchy and has a infinite sequence of conserved quantities

$$\left.\begin{aligned} H_0 &= \frac{1}{2}\int u\, \mathrm{d}x, \quad H_1 = \frac{1}{2}\int u^2\, \mathrm{d}x, \\ H_2 &= \frac{1}{2}\int (2uu_{xx} + u_x^2 - 2u^3)\, \mathrm{d}x, \quad \text{etc.} \end{aligned}\right\} \tag{2.18}$$

Obviously the hierarchy of higher K.d.V. equations, whose Q_K form the conservation laws of the K.d.V. equation itself, can be found by constructing higher operators P in the Lax equation

(1.9*b*). Given an operator L, Gel'fand & Dikii (1976) show how to construct all the hierarchy of operators $P^{(i)}$ that will go with it. The recursion operator relation (2.15) was found by Lenard (see Gardner *et al.* 1974). Magri (1978) has shown how it can be constructed directly from the symmetries of the equation without recourse to the Lax pair. He was also the first to show that the K.d.V. equation has a bi-Hamiltonian structure.

There is an alternative method of finding the conservation laws directly from the transmission coefficient. In the W.K.B. approximation, the reflection coefficient is small for high energies so we shall neglect it and just write the wavefunction $\psi \sim a\exp(-\mathrm{i}kx) = \exp(\mathrm{i}S)$ where $S = \ln a(k, x) - kx$. Substitution of ψ into the Schrödinger equation gives

$$S_x^2 - \mathrm{i}S_{xx} + u = k^2, \quad S_x = -k + (\ln a)_x. \tag{2.19}$$

Expanding $(\ln a)_x$ we have

$$(\ln a)_x = \frac{a_1}{k} + \frac{a_2}{k^2} + \frac{a_3}{k^3} + \dots. \tag{2.20}$$

We find

$$\left.\begin{aligned} &2a_1 - u = 0, \\ &2a_2 + \mathrm{i}a_{1,x} = 0, \\ &2a_3 - a_1^2 + \mathrm{i}a_{2,x} = 0, \quad \text{etc.,} \end{aligned}\right\} \tag{2.21}$$

from which we observe, on recursively calculating the $\int a_i\,\mathrm{d}x$, that the odd ones give the integrals of the K.d.V. equation! This idea can be found in Ablowitz *et al.* (1974).

(*c*) *Other nonlinear equations*

The simplest way to summarize some of the main results on other nonlinear equations is to list some of the results that are analogous to the K.d.V. equation. A matrix generalization of the isospectral idea was first used by Zakharov & Shabat (1972) and generalized further by Ablowitz *et al.* (1974). This involves a Dirac equation form for the spectral problem and we shall make this our first category.

(1) 2×2 *matrix problems*

The spectral problem

$$\left.\begin{aligned} \partial\psi_1/\partial x + \mathrm{i}\lambda\psi_1 = q\psi_2, \\ \partial\psi_2/\partial x - \mathrm{i}\lambda\psi_2 = r\psi_1, \end{aligned}\right\} \tag{2.22}$$

has the 2-channel Schrödinger equation form

$$(-\partial^2/\partial x^2 + \boldsymbol{V})\,\boldsymbol{\psi} = \lambda^2\boldsymbol{\psi}, \tag{2.23a}$$

where the potential matrix is

$$\boldsymbol{V} = \begin{bmatrix} qr & q_x \\ r_x & qr \end{bmatrix}. \tag{2.23b}$$

The following list gives the temporal dependence of the eigenfunctions and the designation of q and r, where

$$\frac{\partial\boldsymbol{\psi}}{\partial t} = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}\boldsymbol{\psi}, \tag{2.24}$$

(i)
$$\left.\begin{aligned} r &= \mp q^*, \\ A &= -2\mathrm{i}\lambda^2 \pm \mathrm{i}\,|q|^2, \\ B &= 2\lambda q + \mathrm{i}q_x, \\ C &= \mp 2\lambda q^* \pm \mathrm{i}q_x^*, \end{aligned}\right\} \quad \mathrm{i}q_t + q_{xx} \pm 2q\,|q|^2 = 0. \tag{2.25}$$

This is the n.l.S. equation, which will be derived in §3. When $r = -q^*$, the spectral problem is skew-adjoint and negative energy eigenvalues (and hence solitons) are possible (see Dr Mollenauer's paper, this symposium, when both cases can occur, depending on the fibre material).

(ii)
$$\left.\begin{aligned} r &= \mp q, \\ A &= -4\mathrm{i}\lambda^3 \pm 2\mathrm{i}\lambda q^2, \\ B &= 4q\lambda^2 + 2\mathrm{i}\lambda q_x \mp 2q^3 - q_{xx}, \\ C &= \mp 4q\lambda^2 \pm 2\mathrm{i}\lambda q_x + 2q^3 \pm q_{xx}, \end{aligned}\right\} \quad q_t \pm 6q^2 q_x + q_{xxx} = 0. \tag{2.26}$$

This is the m.K.d.V. equation.

(iii)
$$\left.\begin{aligned} q &= -r^* = \tfrac{1}{2}R, \\ A &= -S/4\mathrm{i}\lambda, \\ B &= R_t/4\mathrm{i}\lambda, \\ C &= R_t^*/4\mathrm{i}\lambda, \end{aligned}\right\} \quad R_{xt} = RS, \quad S_x = -\tfrac{1}{2}(|R|^2)_t. \tag{2.27}$$

These equations arise in self-induced transparency (s.i.t.) and other areas and contain the sine–Gordon equation: when R is restricted to be real, let $R = \varphi_x$ and $S = \pm\cos\varphi$, then (2.27) integrates to
$$\varphi_{xt} = \pm\sin\varphi. \tag{2.28}$$

(iv)
$$\left.\begin{aligned} q &= r^* = A_2, \\ A &= -|A_1|^2/2\mathrm{i}\lambda, \\ B &= A_1^2/2\mathrm{i}\lambda, \\ C &= -A_1^{*2}/2\mathrm{i}\lambda, \end{aligned}\right\} \quad A_{1,x} = A_2 A_1^*, \quad A_{2,t} = A_1^2. \tag{2.29}$$

Equations (2.29) are the second harmonic resonance equations (see Kaup 1978) mentioned in (3.18).

(2) 3×3 *matrix problems*

(i) Given the Lax pair (Yajima & Oikawa 1976; Ma 1978)
$$\mathrm{L} = \begin{bmatrix} \tfrac{1}{3}\partial/\partial x & -\tfrac{1}{3}A & -\tfrac{1}{3}\mathrm{i}B \\ 0 & \tfrac{1}{2}\partial/\partial x & -\tfrac{1}{2}A^* \\ \mathrm{i} & 0 & \partial/\partial x \end{bmatrix}, \tag{2.30}$$
$$\mathrm{P} = \begin{bmatrix} \mathrm{i}\lambda^2 & 2(\lambda A - \mathrm{i}A_x) & -2\mathrm{i}\,|A|^2 \\ -2A^* & -\mathrm{i}\lambda^2 & 2(\mathrm{i}A_x^* - \lambda A^*) \\ 0 & -2A & \mathrm{i}\lambda^2 \end{bmatrix}. \tag{2.31}$$

Then A and B evolve according to

$$iA_t - 2A_{xx} + 2AB = 0, \quad B_t = -4(|A|^2)_x, \tag{2.32}$$

which are the long-wave–short-wave resonance equations (see (3.24) and (3.25)).

(ii) *The three-wave resonance equations* written in (3.16) with all $\mu_i^2 = 1$ have a decay instability when the sign of one of the μs differs from the other two, while for the explosive instability the signs of the μs must be the same. Zakharov & Manakov (1976) showed the isospectral deformation was of third order, but Kaup (1976) has shown that in the limit $t \to \pm\infty$ this third-order problem can be factored into three separate second-order Zakharov–Shabat-type second-order problems. The scattering problem is complicated to write down and the reader is referred to the above references for details.

(3) *Scalar Lax equations of degree three*

(i) Let

$$\left.\begin{aligned} L &= 4D^3 + (1+6w_x)\,D + 3(w_{xx} - i\surd 3 w_t), \quad (2.33)\\ P &= i\surd 3(D^2 + w_x), \quad (2.34)\end{aligned}\right\} \quad D \equiv \partial/\partial x,$$

then we find that w satisfies

$$(w_{xxx} + 3w_x^2 + w_x)_x - w_{tt} = 0. \tag{2.35}$$

When $u = w_x$ then (2.35) becomes

$$u_{xxxx} + 6(uu_x)_x + u_{xx} - u_{tt} = 0. \tag{2.36}$$

This is the so-called Boussinesq equation, which is a 2-way version of the K.d.V. equation. This Lax pair was found by Zakharov (1974).

(ii) Take first the pair of operators

$$L_1 = D^3 + uD, \tag{2.37}$$

$$P_1 = 9D^5 + 12uD^3 + 15u_x\,D + (5u^2 + 10u_{xx})\,D, \tag{2.38}$$

which give

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x\,u_{xx} + 5u^2u_x, \tag{2.39}$$

and then take

$$\left.\begin{aligned} L_2 &= D^3 + 2wD + w_x\\ P_2 &= 9D^5 + 30wD^3 + 45w_x\,D^2 + (20w^2 + 35w_{xx})\,D + (10w_{xxx} + 20ww_x),\end{aligned}\right\} \tag{2.40}$$

which give

$$w_t = w_{xxxxx} + 10ww_{xxx} + 25w_x\,w_{xx} + 20w^2w_x. \tag{2.41}$$

Equations (2.39) and (2.41) seem totally unrelated and were derived by Sawada & Kotera (1974) and B. Kupershmidt (personal communication, 1979), respectively, as the two canonical third-order scalar spectral problems. Fordy & Gibbons (1980) have shown in fact that they are related in the following way:

$$\left.\begin{aligned} L_1 &= (D-v)\,(D+v)\,D,\\ L_2 &= (D+v)\,D(D-v),\end{aligned}\right\} \tag{2.42}$$

where u and w are connected to the new variable v by the Miura transformations

$$\left.\begin{aligned} u &= v_x - v^2, \\ w &= -v_x - \tfrac{1}{2}v^2, \end{aligned}\right\} \tag{2.43}$$

and v satisfies another fifth-order equation. These three fifth-order equations in u, w and v, which are connected via (2.43), are the three equations associated with third-order scalar Lax problems in which only one variable occurs.

(4) *Discrete equations*

Flaschka (1974) and Manakov (1975) independently extended the idea of the Lax commutator bracket $L_t = [P, L]$ to matrices. In the periodic case ($Q_n = Q_{n+N}$), if we choose

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & & 0 \\ 0 & a_2 & b_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_N & \cdots & & \cdots & b_N \end{bmatrix}, \tag{2.44a}$$

$$P = \begin{bmatrix} 0 & a_1 & 0 & \cdots & -a_N \\ -a_1 & 0 & a_2 & & \cdot \\ 0 & -a_2 & 0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_N & \cdots & & \cdots & 0 \end{bmatrix}, \tag{2.44b}$$

we find that the *a*s and *b*s are related by

$$\left.\begin{aligned} \dot{b}_n &= 2(a_n^2 - a_{n-1})^2, \\ \dot{a}_n &= a_n(b_{n+1} - b_n). \end{aligned}\right\} \tag{2.45}$$

Taking $b_n = -\frac{1}{2}\dot{Q}_{n-1}$ and $a_n = \frac{1}{2}\exp[\frac{1}{2}(Q_{n-1} - Q_n)]$, we find that the Qs satisfy

$$\ddot{Q}_n = \exp(Q_{n-1} - Q_n) - \exp(Q_n - Q_{n+1}), \tag{2.46}$$

which is the Toda lattice. Moser (1975) has shown that N unit mass particles under an inverse square repulsive potential on the line is also a system solvable by choosing appropriate matrices for L and P.

I have effectively given a list of the isospectral problems that solve various nonlinear equations, although I have given only a crude sketch of how the inverse problem can be solved for the scalar Schrödinger operator (2.1*a*). The other spectral problems are more complicated to solve although the Zakharov–Shabat scheme has a certain simplicity because of the elegance of its formulation. Ablowitz *et al.* (1974) considered this in detail. For higher order spectral problems such as (2.33) and (2.37), Caudrey (1982) has shown how to solve an $N \times N$ spectral problem of the form

$$\partial\boldsymbol{\varphi}/\partial x = [A(\lambda) + B(x, \lambda)] \cdot \boldsymbol{\varphi}, \tag{2.47}$$

where A and B are $n \times n$ matrices. The Boussinesq problem (2.35) comes into this category, for example.

3. *Classification of wave motions and integrable systems*

To describe which classes of problems in applied mathematics and theoretical physics yield integrable systems we must first look at different possible types of wave motions and wave resonance interactions. Nonlinear dispersive media can support many different types of wave motion. There are several canonical types, which occur in a variety of physical problems. Instead of working through long and complicated particular examples I will endeavour to show how these canonical types can arise by working from a general class of equation from the beginning. These general classes of equations may not contain all possible physical examples, but they can always be modified easily enough. Specific problems can be found in the references. There are exceptions and special cases of the generalities discussed here, of course, but we will mainly concentrate on the general classifications, rather than specific examples.

(*a*) *Long waves*

Let us assume our full system of equations has a real dispersion relation

$$\omega = \omega(k) \tag{3.1}$$

so there are no loss terms, and let ϵ, a small parameter, be a measure of the amplitude of the various dependent variables

$$\boldsymbol{u} = \epsilon \boldsymbol{u}^{(1)}(x,t) + \epsilon^2 \boldsymbol{u}^{(2)}(x,t) \dots . \tag{3.2}$$

We now need to look for the variation of $\boldsymbol{u}^{(1)}(x,t)$ over some long length scale that is connected with ϵ. The problem can be looked at simply in this way. For long wavelengths, we need small wavenumbers k and to relate these to ϵ we write $k = \epsilon^p \kappa$, where κ is a new wavenumber of $O(1)$ on a long scale and p is an index yet to be determined. Since the systems we shall consider will be dispersive, $\omega(k)$ will contain either all odd or all even terms in k. We will choose odd and so we can write an expression for $\omega(k)$ as

$$\omega(k) = ak + bk^3 + \dots \tag{3.3}$$

for small k. For travelling waves with argument $\theta = kx - \omega(k)\, t$ we can now rewrite this as

$$\theta = \epsilon^p (x - at)\,\kappa - b\kappa^3(\epsilon^{3p} t), \tag{3.4}$$

which indicates that we can take new 'stretched' coordinates

$$\xi = \epsilon^p (x - at), \quad \tau = \epsilon^{3p} t. \tag{3.5}$$

Most problems in plasmas and fluids have their basic equations of motion written in continuity form through equations of conservation of mass and momentum, etc. As a simple problem, which we shall take as an archetype, let us consider a system of equations of the form

$$\mathrm{L}\left(\frac{\mathrm{D}}{\mathrm{D}t}; \frac{\partial}{\partial x}\right) u = 0, \tag{3.6a}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}. \tag{3.6b}$$

We shall leave (3.6) in scalar form although the same problem can be generalized to matrices. L is some linear operator that is polynomial in $\mathrm{D}/\mathrm{D}t$ and $\partial/\partial x$. While not every system can be put in this class, it will serve as the simplest model on which to do reductive perturbation theory. Since the dispersion relation is

$$\mathrm{L}(-\mathrm{i}\omega;\mathrm{i}k) = 0, \tag{3.7}$$

which must be real, we will take it as odd. Using new coordinates (ξ, τ) in (3.5) and Taylor expanding (3.6*a*), we find to the three lowest orders in ϵ:

$$\left\{\mathrm{L}_1\left[\epsilon^{3p}\frac{\partial}{\partial\tau}-a\epsilon^{p}\frac{\partial}{\partial\xi}+\epsilon^{p+1}u^{(1)}\frac{\partial}{\partial\xi}\right]+\mathrm{L}_2\,\epsilon^{p}\frac{\partial}{\partial\xi}+\mathrm{L}_{222}\,\epsilon^{3p}\frac{\partial^3}{\partial\xi^3}+\ldots\right\}(\epsilon u^{(1)}+\ldots) = 0. \tag{3.8}$$

The suffixes 1 and 2 mean differentiation with respect to the first and second positions in L evaluated at $(0,0)$. The parameter a is specified by the equation $a\mathrm{L}_1 = \mathrm{L}_2$. This removes the lowest order term. To match the terms at $3p+1$ and $p+2$ we obviously require $p = \frac{1}{2}$ and we are then left, at $O(\epsilon^{\frac{5}{2}})$, with

$$\mathrm{L}_1(u^{(1)}_{\tau}+u^{(1)}\,u^{(1)}_{\xi})+\mathrm{L}_{222}\,u^{(1)}_{\xi\xi\xi} = 0, \tag{3.9}$$

which is the K.d.V. equation. The value $p = \frac{1}{2}$ is universal for the K.d.V. equation. There are more complicated classes of problem other than (3.6) but the scalings and the details turn out essentially the same. The K.d.V. equation can be thought of as the typical equation that governs the evolution of long shallow waves in the (ξ, τ) coordinates in quadratically nonlinear systems. If the problem has cubic nonlinearity at lowest order then we would find $p = 1$ and obtain the m.K.d.V. equation. There are an enormous number of references in which the K.d.V. or m.K.d.V. equations have been derived. Korteweg & de Vries (1895) were the first in studying the problem of surface waves on shallow water. Taniuti & Wei (1968) and Su & Gardner (1969) used the reductive perturbation technique to show generally how K.d.V. occurs, although Washimi & Taniuti (1966) derived the K.d.V. equation earlier for ion acoustic waves. Many of the appropriate references, particularly in plasma physics, can be found in the review by Scott *et al.* (1973) and Ichikawa (1979). The books by Karpman (1975) and Dodd *et al.* (1982) contain an appropriate list of references. The study of the K.d.V. equation as an initial value problem is clearly very important physically as well as mathematically.

(*b*) *Wave packets*

In (*a*) we found that the K.d.V. equation governs the motion of long 'pulses' in dispersive systems. If instead we want to know how *oscillations* behave, whose wavelength is much shorter, we need a different method. An applied mathematician would use the method of multiple scales, but a physicist would use a rough form of slowly varying envelope approximation. Both give the same answer: namely an evolution equation for the slowly varying amplitude of a packet of oscillations.

If the original equations take the form of (3.6)

$$\mathrm{L}\left(\frac{\partial}{\partial t};\frac{\partial}{\partial x}\right)\varphi = \beta\varphi^3 \tag{3.10}$$

say, then the method of multiple scales requires the introduction of 'slow' scales.

$$\bar{X} = \epsilon(x-c_{\mathrm{g}}\,t); \quad T = \epsilon^2 t, \tag{3.11}$$

where φ or u are expanded as

$$\varphi = \epsilon\varphi^{(1)}+\epsilon^2\varphi^{(2)}+\ldots. \tag{3.12}$$

The full method can be found in any textbook (see Nayfeh 1973). We look for the amplitude $A(\bar{X}, T)$ of an oscillating solution at $O(\epsilon)$ and obtain

$$\varphi^{(1)} = A(\bar{X}, T) \exp(\mathrm{i}\theta) + \mathrm{c.c.} \tag{3.13}$$

with $\mathrm{L}(-\mathrm{i}\omega; \mathrm{i}k)$ as the dispersion relation, where $\theta = kx - \omega t$ and $c_g = \mathrm{d}\omega/\mathrm{d}k$. At $O(\epsilon^2)$ no secular terms appear due to the choice of the X and T scales in (3.11). At $O(\epsilon^3)$, when secular terms are removed we find that A must satisfy

$$2\mathrm{i}\frac{\partial A}{\partial T} + \left(\frac{\partial^2 \omega}{\partial k^2}\right)\frac{\partial^2 A}{\partial \bar{X}^2} + \gamma A |A|^2 = 0, \quad \gamma \text{ real.} \tag{3.14}$$

The coefficient of the second derivative can be shown to be $\partial^2\omega/\partial k^2$ by using the second total k-derivative of the dispersion relation. This derivation of the n.l.S. equation still holds for more complicated right-hand sides of (3.10) provided the nonlinearity is either quadratic or cubic in φ and its derivatives.

Hence we find that where the K.d.V. or m.K.d.V. equations arise generally for *long* waves, the n.l.S. equation arises for slowly varying oscillating waves. The model problem (3.10) actually arises in optics, where L is derived from Maxwell's equations, φ is the field E and the cubic term arises as a nonlinear term in the polarization resulting from the refractive index of the medium being dependent on the field at higher powers,

$$n = n_0 + n_2 |E|^2. \tag{3.15}$$

The sign of n_2 will determine the sign of γ in (3.14). Dr Mollenauer's paper (this symposium) is, in a sense, based on this phenomenon. An optical fibre consists of a silica core whose refractive index behaves as in (3.15). For certain frequencies, $\gamma(\partial^2\omega/\partial k^2) < 0$ and no solitons are possible, only self-similar solutions. For other frequencies $\gamma(\partial^2\omega/\partial k^2) > 0$ and soliton production is possible from an initial pulse fed into the fibre. The papers on the n.l.S. equation are even more numerous than the K.d.V. equation, both in plasma physics, fluid dynamics and optics: see references in Scott *et al.* (1973), Benney & Newell (1967), Taniuti & Yajima (1969, 1973), Tanuiti & Wei (1968), Ichikawa (1979), Newell (1974), Stuart & DiPrima (1978), Karpman (1975), Chu & Mei (1970, 1971) and Dodd *et al.* (1982). For the place of the n.l.S. in water wave problems, the papers by Hasimoto & Ono (1972) and Peregrine (1983) make interesting reading. For the optical work, see Dr Mollenauer's paper from this symposium. Indeed the n.l.S. equation has a different life in solid state physics and vortex dynamics since it arises in Heisenberg spin chains and problems in vortex stretching (Hasimoto 1972).

Cases where these scalings break down usually arise for values of the wavenumber when a singularity appears in γ. This occurs at second harmonic resonance and the long-wave–short-wave resonance in capillary gravity waves (Djordjevic & Redekopp 1977).

(*c*) *Wave–wave interactions*

Many problems in physics and applied mathematics are not quite so simple as case (*b*), where only one frequency arises. Electrical engineers, laser physicists and fluid dynamicists, to name but three groups, know well that 'triad' resonances can occur if a system is pumped with one frequency that can be tuned such that a 'signal' wave and an 'idler' form a resonance condition $k_1 + k_2 = k_3$ and $\omega_1 + \omega_2 = \omega_3$. If this happens then so-called 3-wave mixing or resonance occurs.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This is a *quadratic* resonance phenomenon, whereas the n.l.S. occurs as a cubic resonance, and will occur in most systems of the type

$$\mathscr{L}\left(\frac{\partial}{\partial t};\frac{\partial}{\partial x}\right)\varphi = \alpha\varphi^2 \tag{3.15}$$

or even in (3.6). Using the scales $X_1 = \epsilon x$, $T_1 = \epsilon t$ and taking

$$\varphi^{(1)} = A_1\,e^{i\theta_1}+A_2\,e^{i\theta_2}+A_3\,e^{i\theta_3}+\text{c.c.} \tag{3.16}$$

with $\theta_1+\theta_2 = \theta_3$, we find that removal of secular terms of $O(\epsilon^2)$, using the multiple scales method on (3.15), gives

$$\left(\frac{\partial}{\partial T_1}+c_1\frac{\partial}{\partial X_1}\right)A_1 = \gamma_1 A_3 A_2^*, \tag{3.16a}$$

$$\left(\frac{\partial}{\partial T_1}+c_2\frac{\partial}{\partial X_1}\right)A_2 = \gamma_2 A_1^* A_3, \tag{3.16b}$$

$$\left(\frac{\partial}{\partial T_1}+c_3\frac{\partial}{\partial X_1}\right)A_3 = \gamma_3 A_1 A_2. \tag{3.16c}$$

The behaviour in (3.16) is very different depending on the γ_i. Kaup *et al.* (1979) have made a massive review of this phenomenon. As mentioned in §2 these are solvable by the inverse scattering transform, as they also are in several dimensions (Kaup 1980; see also Craik 1978). Note that they are non-dispersive equations even though they were originally derived from a dispersive system. The 3-wave mixing phenomenon is so common in physics that it can be found almost everywhere: in Raman scattering in crystals, laser cavity devices, electric circuits, water-wave tanks and plasma problems.

A special case of these is second harmonic resonance. This usually occurs when it is possible to achieve $2k_1 = k_2, 2\omega_1 = \omega_2$ for two wavenumbers k_1 and k_2. Similarly to (3.16), when

$$\varphi^{(1)} = A_1\,e^{i\theta_1}+A_2\,e^{i\theta_2}+\text{c.c.} \tag{3.17}$$

we find

$$\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)A_1 = A_2 A_1^*, \tag{3.18a}$$

$$\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)A_2 = A_1^2. \tag{3.18b}$$

This phenomenon is harder to achieve since the condition $c_p(2k_1) = c_p(k_1)$ will only occur in systems whose dispersion relations allow this strange condition. They usually need to be multiply branched (see figure 3*a*) or of the type in figure 3*b*.

The interaction of capillary waves and gravity waves in water displays this phenomenon because the dispersion relation (with surface tension) is of the type shown in figure 3*b* (see McGoldrick (1980) and Djordjevic & Redekopp (1977)). Again, as we saw in §2, equations (3.18) are completely integrable in a characteristic coordinate frame.

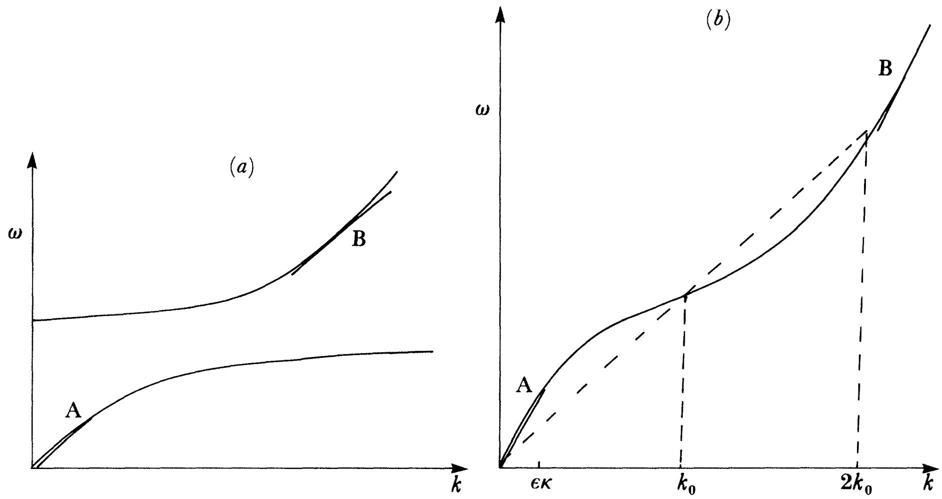

FIGURE 3. (*a*) The long-wave–short-wave resonance depicted on a double-branched dispersion relation. The secant c_p at A is parallel to the tangent c_g at B. (*b*) A typical dispersion curve showing second harmonic resonance at k_0 and $2k_0$. The points A and B denote the long-wave–short-wave resonance with the secant at A parallel to the tangent at B ($c_p = c_g$).

(*d*) *Coupling of long waves and short waves*

The paper by Dr Scott, in this symposium, on a possible mechanism for the propagation of energy down α-helix-type long-chain molecules is mathematically based on a very interesting phenomenon in wave motion which has a more general application. To illustrate the idea I will construct a mechanical model that consists of a 'guitar string' that can support both transverse waves, which occur when the string is plucked, and longitudinal compressive waves down the string (see figure 4).

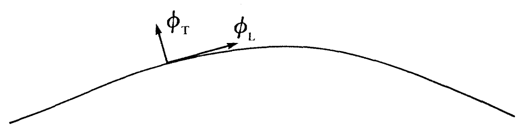

FIGURE 4. Guitar string with both longitudinal and transverse propagating waves.

We shall let φ_T and φ_L be the transverse and longitudinal displacement respectively. Without any transverse motion we would expect lossless, dispersive longitudinal waves to propagate down the string with a dispersion relation $L(-i\omega; ik) = 0$, arising from the linear problem

$$L\left(\frac{\partial}{\partial t}; \frac{\partial}{\partial x}\right)\varphi_L = 0. \tag{3.19}$$

The form of L depends, of course, on the string. When the string is plucked transversely, these longitudinal waves have to propagate along a curved string, rather like light rays curving in a medium that has a varying refractive index. We therefore think of the dispersion relation as being dependent on φ_t

$$L\left(\frac{\partial}{\partial t}; \frac{\partial}{\partial x}; \varphi_t\right)\varphi_\ell = 0. \tag{3.20}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Taking a slowly varying oscillating solution for φ_ℓ,

$$\varphi_L = \epsilon A(x,t) \exp(i\theta) + \text{c.c.} \tag{3.21}$$

as for the n.l.S. equation, and matching of the scales shows that a long wave (not an oscillating wave) is the dominant behaviour in φ_t

$$\varphi_T \sim \epsilon^2 n(x,t) + \dots. \tag{3.22}$$

We expect this because any oscillations from (3.21) would be too fast for φ_t to react. We can therefore think of guitar string behaviour modelled by a wave equation that is driven by the displacement in the string caused by the fast oscillations in the longitudinal mode

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_p^2}\frac{\partial^2}{\partial t^2}\right) n = -\beta \frac{\partial^2}{\partial x^2}(|A|^2). \tag{3.23}$$

The term on the right side can be worked out dimensionally via the fact that the driving force is the derivative of the intensity ($|A|^2$) of the longitudinal forcing. Equations (3.20) and (3.23) would then be the two basic equations governing the coupling of transverse and longitudinal waves. Note that the only nonlinearity comes from the coupling between φ_T and φ_L: we have not assumed that the string is intrinsically nonlinear.

In plasma physics this phenomenon models the interaction between Langmuir waves (electron oscillations), represented by the longitudinal mode, and ion sound, represented by the transverse mode. The form of L turns out, in this case, to be

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + nA = 0 \tag{3.24}$$

and (3.23) and (3.24) are known as Zakharov's (1972) equations.

Exactly the same equations arise in Davydov's model (Davydov 1979; Davydov & Kislukha 1976), which Scott discusses in his paper. In this case, the transverse mode represents phonons (elastic sound waves) and the longitudinal mode represents the energy that disperses along the chain by a dipole–dipole interaction due to an infrared resonance in the carbon–oxygen double bond of the amide group. Scott's equations are normally expressed in discrete form, but Zakharov's equations are the continuous form of the model. The fundamental point about Zakharov's equations is that in the limit when c_p is taken to be large, which expresses the fact that the second-order time dependence of n is ignored, then we find that n is proportional to $|A|^2$ and (3.24) becomes the n.l.S. equation and soliton behaviour occurs.

Another interesting limit, which we do not have space to discuss in detail, is the one-way version of Zakharov's equations, where (3.23) is replaced by

$$n_x + n_t/c_p = -\beta_1(|A|^2)_x. \tag{3.25}$$

Equation (3.25), coupled to (3.24), is now an integrable system which, as we saw in §2, has a 3×3 Lax pair. These integrable equations arise on different scales, usually $\bar{X} = \epsilon(x - c_g t)$; $T_2 = \epsilon^2 t$, scales of the n.l.S. equation. One finds that two limits exist; either the n.l.S. limit or the limit where n and $|A|$ satisfy (3.25) in the $\bar{X}$, T coordinates under the condition that $c_p = c_g$. This phenomenon is known as the long-wave–short-wave resonance and was first discussed by Benney (1976). It occurs in water waves in the interaction of gravity and capillary waves (see Djordjevic & Redekopp (1977)) and also in internal waves (Grimshaw 1977). A full discussion is given in the paper by Elgin *et al.* (1985). Suffice to say that this resonance mechanism is a limit of the 3-wave resonance: $k_1 = k_2 + k_3$; $\omega_1 = \omega_2 + \omega_3$. Think of k_1 and k_2 being sidebands of a main wave k, such that $k_1 = k + \epsilon\kappa$, $k_2 = k - \epsilon\kappa$ and $k_3 = 2\epsilon\kappa$, where κ is $O(1)$. To achieve the triad condition for the ωs, we expand: $\omega(k+\epsilon\kappa) - \omega(k-\epsilon\kappa) = \omega_3$, to find

$$2\epsilon\kappa \, \mathrm{d}\omega/\mathrm{d}k = \omega_3$$

to $O(\epsilon)$. Hence this is satisfied provided the group velocity c_g of the short wave (k) equals the phase velocity of the long wave $\epsilon\kappa$. The dispersion curves in figure 3*a*, *b* show that this condition is easily possible under these circumstances. Indeed both the long-wave–short-wave resonance *and* second harmonic resonance occur in capillary gravity waves (but not together!). This usually shows when a singularity occurs in the coefficient of $A\,|A|^2$ in the n.l.S. equation, which indicates that a rescaling is necessary. In water waves the scales are $\overline{X} = \epsilon^{\frac{2}{3}}(x - c_g t)$, $T = \epsilon^{\frac{4}{3}} t$ for the long-wave–short-wave process.

(*e*) *Dispersive systems not in equilibrium*

We have seen in (*b*) that, in general, the n.l.S. equation governs the motion of the amplitude of wave packets in stable dispersive systems. There is no energy loss nor energy absorption. There are some physical problems where the system has a source of available potential energy (a.p.e.) even though it remains dispersive. A bifurcation problem then arises in which the character of the problem and the scales change near the bifurcation point. Stuart's method (Stuart 1960) of using multiple scales near the critical point gives the amplitude equations, which operate on a faster timescale than the n.l.S. equation. We need $X = \epsilon x$, $T = \epsilon t$ and find that the amplitude equations become

$$\left(\frac{\partial}{\partial T} + c_1 \frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T} + c_2 \frac{\partial}{\partial X}\right) A = \alpha A - \beta A B, \tag{3.26a}$$

$$\left(\frac{\partial}{\partial T} + c_2 \frac{\partial}{\partial X}\right) B = \left(\frac{\partial}{\partial T} + c_1 \frac{\partial}{\partial X}\right) |A|^2, \tag{3.26b}$$

where c_1 and c_2 are the two group velocities that occur at the critical point. Equations (3.26) are more recognizable if we make the transformations

$$R = \surd 2A, \quad S = \pm 1 - \beta\alpha^{-1} B, \tag{3.27a}$$

$$\xi = -\alpha^{-\frac{1}{2}}(X - c_1 T)\,(c_1 - c_2)^{-1}, \quad \tau = \alpha^{\frac{1}{2}}(X - c_2 T)\,(c_1 - c_2)^{-1}. \tag{3.27b}$$

They become

$$R_\tau = \mathscr{P}, \tag{3.28a}$$

$$\mathscr{P}_\xi = RS, \tag{3.28b}$$

$$S_\xi = -\tfrac{1}{2}(R^* \mathscr{P} + R \mathscr{P}^*). \tag{3.28c}$$

These equations occur both in self-induced transparency (s.i.t.) (Lamb 1971) and two problems in baroclinic wave motion (Gibbon *et al.* 1979). The sine–Gordon equation

$$\varphi_{\xi\tau} = \pm \sin\varphi \tag{3.29}$$

is embedded in (3.26) when R is taken to be real ($R = \varphi_\xi$, $S = \pm\cos\varphi$, $\mathscr{P} = \pm\sin\varphi$). The paper by Gibbon & McGuinness (1981) discusses this type of general instability.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

4. The connection of integrable p.d.es with Lie algebras

To the applied mathematician or physicist who is more used to 'classical' mathematics, the considerable portion of this symposium devoted to Lie algebras must come as a surprise. In an attempt to explain how they occur I must necessarily take an exceedingly brief and naïve overview of *how* they are connected with soliton equations. *Why* the two are connected in a deep sense is something I cannot hope to answer here; the specialist articles on this topic will give a better answer. For the non-specialist who wants to see exactly how a Lie algebra occurs, I will again use the K.d.V. equation as an example.

Let us assume that we are looking for an isospectral problem for our given p.d.e. In the most general form, we will write it as an $n \times n$ problem

$$\boldsymbol{\psi}_x = F(u;\lambda)\,\boldsymbol{\psi}, \tag{4.1}$$

where $\boldsymbol{\psi}$ is an n-component eigenfunction. A constant λ will come in at some point and will act as an eigenvalue. We are naturally working on the assumption that we do not know the form of (4.1) for the K.d.V. equation, which in reality is a 2×2 version of (4.1) with F linear in u and λ. For the K.d.V. equation with three space derivatives, a matching form for the time evolution of $\boldsymbol{\psi}$ is

$$\boldsymbol{\psi}_t = G(u, u_x, u_{xx})\,\boldsymbol{\psi}. \tag{4.2}$$

Our task is to find the forms of F and G when u satisfies the K.d.V. equation. Cross differentiation gives

$$F_t = G_x + [G, F] \tag{4.3}$$

where $[G,F] = GF - FG$. From now on we shall just assume the G and F are $n \times n$ matrices. Our first assumption is that F is a function of u only. Substituting for u_t from the K.d.V. equation itself, we find

$$(6uu_x - u_{xxx})\,F_u = u_x\,G_u + u_{xx}\,G_{u_x} + u_{xxx}\,G_{u_{xx}} + [G,F]. \tag{4.4a}$$

This now becomes an algebraic problem in u and its derivatives. Matching coefficients of u_{xxx} in (4.4a) we find

$$G_{u_{xx}} = -F_u \quad \text{so} \quad G = -u_{xx}\,F_u + \alpha(u, u_x). \tag{4.4b}$$

Using this in the remnant of (4.4a) we obtain

$$6uu_x\,F_u = [\alpha, F] + u_{xx}\{\alpha_{u_x} - [F_u, F] - u_x\,F_{uu}\} + u_x\,\alpha_u. \tag{4.5}$$

Since neither F nor α contain u_{xx}, its coefficient must be zero, thereby giving

$$\alpha = \tfrac{1}{2}u_x^2\,F_{uu} + u_x[F_u, F] + \beta(u); \tag{4.6}$$

$\beta(u)$ can then be determined from the remnant of (4.5) and the process continued until we obtain a set of conditions upon $F(u)$ from the various coefficients of u_x^3, u_x^2, etc. These are

$$F_{uuu} = 0, \tag{4.7a}$$

$$[F_u, F]_u + \tfrac{1}{2}[F_{uu}, F] = 0, \tag{4.7b}$$

$$\beta_u = 6uF_u - [[F_u, F], F], \tag{4.7c}$$

$$[\beta, F] = 0. \tag{4.7d}$$

Equation (4.7 *a*) can be solved immediately to give

$$F = X_1 + X_2 u + X_3 u^2, \tag{4.8}$$

where X_1, X_2 and X_3 are constants of integration, which are matrices of, as yet, unknown dimension.

In the rest of (4.7) we find

$$[X_1, X_3] = [X_2, X_3] = 0, \tag{4.9a}$$

$$\beta = 4u^3 X_3 + 3u^2 X_2 - \tfrac{1}{3}u^3[X_3, [X_1, X_2]] - \tfrac{1}{2}u^2[X_2, [X_1, X_2]] - u[X_1[X_1 X_2]] + X_4. \tag{4.9b}$$

Defining $[X_1, X_2] = -X_7, [X_1, X_7] = X_5$ and $[X_2, X_7] = X_6$, $[\beta, F] = 0$ now gives, for each coefficient of powers of u,

$$\left.\begin{aligned} &[X_1, X_4] = 0 = [X_3, X_6], \\ &[X_2, X_4] + [X_1, X_5] = 0, \quad [X_3, [X_3, X_7]] = 0, \\ &\tfrac{3}{2}[X_1, X_6] + [X_3, X_4] = 3X_7, \\ &\tfrac{1}{2}[X_2, X_6] + [X_3, X_5] + \tfrac{1}{3}[X_1, [X_3, X_7]] = 0. \end{aligned}\right\} \tag{4.10}$$

The bracket relations $[X_i, X_j] = \alpha_{ij} X_K$ constitute a Lie algebra. Further brackets can be found from the Jacobi identities. If a representation for this algebra can be found, i.e. matrices X_i of known dimension can be found that satisfy the commutation relations above, then we have found $F(u)$ and $G(u, u_x, u_{xx})$ exactly and hence the spectral problem. For those whose primary interest is finding the spectral problem for a given p.d.e., this method is very useful. Dodd & Fordy (1983) have worked out a set of rules that can be used to find an algebra and then find its representation. This method also has the advantage of failing quite spectacularly when the equation is not integrable. For instance, if we take a K.d.V.-type equation with a $u^3 u_x$ nonlinearity then the algebra is empty and no spectral problem exists.

The classification of the type of algebra is important. For most of those equations mentioned in §2, the algebras obtained will be copies of sl $(2, \mathbb{R})$ or, more generally, sl $(2, \mathbb{C})$ (2×2 traceless matrices). The standard basis for sl $(2, \mathbb{C})$ is

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{4.11a}$$

which satisfy the commutation relations

$$[h, e_\pm] = \pm 2e_\pm, \quad [e_+, e_-] = h. \tag{4.11b}$$

The general method described above was first put forward by Wahlquist & Estabrook (1975), who used the language of differential geometry. Since then Dodd & Fordy (1983) have turned the method into more of an algorithm for finding spectral problems and give various rules and methods for finding the representation for the algebra. If sl $(2, \mathbb{C})$ does not work, then one must try sl $(3, \mathbb{C})$, such as in the case of the Boussinesq equation or the long-wave–short-wave equations. For the K.d.V. algebra given above we first notice that since the K.d.V. equation has the scale symmetry $x \to \lambda^{-1}x, t \to \lambda^{-3}t, u \to \lambda^2 u$, then the algebra can be rescaled in a similar way, for example $\hat{X}_1 = \lambda X_1, \hat{X}_2 = \lambda^{-1}X_2$, etc. Using this fact and using the various rules in algebra for identifying nil-potent matrices one can easily find

$$F = -e_+ + (\lambda^2 - u)\, e_- \tag{4.12}$$

and a similar expression for G. These both require a certain amount of effort in identifying the elements X_i in terms of the basis of $\mathrm{sl}(2, \mathbb{C})$. Equation (4.12) is now

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}_x = \begin{bmatrix}0 & u-\lambda^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{4.13a}$$

which becomes, on the elimination of ψ_1,

$$(-\partial^2/\partial x^2 + u)\,\psi_2 = \lambda^2\psi_2. \tag{4.13b}$$

Equation (4.13*b*) is exactly our Schrödinger spectral problem. Although this result was known anyway, the method described above has been successful in finding previously unknown spectral problems (Dodd & Gibbon 1977; Dodd & Fordy 1982, 1983) and will yield all the standard ones. It is therefore useful as a test of integrability.

The above has been a very crude and brief description of how these equations are connected with Lie algebras. The much deeper problem, which is addressed by the pure mathematicians in the subject, really turns the problem on its head. One can argue that these algebras are the *fundamental* objects and classification of these will give the global picture with regard to integrable systems. It is in this direction that much work has been directed, particularly along the line of so-called Kac–Moody algebras. The Kac–Moody extension of an algebra, in crude terms, is to take a Laurent polynomial a, say,

$$a = \sum_{i=-N}^{M} a_i z^i, \tag{4.14}$$

whose coefficients belong to the algebra in question, i.e. in this case $a_i \in \mathrm{sl}(2, \mathbb{C})$. These Kac–Moody extensions can be graded by choosing appropriate powers of z. The papers by Dr Wilson and Professor Frenkel, in this symposium, contain a discussion of this approach.

5. Integrability and the Painlevé method

Most of the ideas expressed in the papers in this symposium, with the exception of Professor van Moerbeke's, have been appropriate for either p.d.es or differential difference equations like the Toda lattice. The idea of integrability was discussed by Euler, Lagrange, Poincaré and others in their work in mechanics. The motion of a rigid body is a famous example from the last century of a problem that is only integrable in special cases. The Euler–Poisson equations are six coupled, first-order differential equations in three components of angular velocity and three angles. The moments of inertia (A, B, C) and the centre of gravity coordinates are adjustable. There are three standard integrals (mass, energy and momentum), and given standard reductions, the problem reduces to finding a fourth integral.

Kovalevskaya (1890) studied this problem by looking for a fourth integral via the singularities of the equation. She found that movable poles only exist for four special cases: the three known ones ($A = B = C; x_0 = y_0 = z_0 = 0; A = B, x_0 = y_0 = 0$) and a new fourth one ($A = B = 2C, z_0 = 0$). The fourth integral was indeed identified in this new case. To look for movable poles in complex time one needs to prove the existence of a Laurent series

$$X(t) = (t-t_0)^{-n} \sum_{m=0}^{\infty} a_m (t-t_0)^m. \tag{5.1}$$

The value of n at leading order, which must be integer, is easily determined by the problem under study. The best example of this is the Hénon–Heiles (1964) system

$$\ddot{x} = -Ax - 2Dxy, \tag{5.2a}$$

$$\ddot{y} = -By - Dx^2 + Cy^2. \tag{5.2b}$$

For $\lambda = D/C$, Chang *et al.* (1982) found that this method yields only four values, $\lambda = -1$, $-\frac{1}{2}$, $-\frac{1}{6}$, $-\frac{1}{16}$, for which a Laurent series exists with integer leading order, and they were indeed able to identify a new fourth integral for the $\lambda = -\frac{1}{16}$ case. This method is intimately connected with that of Painlevé (see Ince 1956), who studied equations of the type

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = F\left(\frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right),$$

where F is analytic in x and rational in y and $\mathrm{d}y/\mathrm{d}x$. He found 50 types whose only movable singularities were poles, 44 of which have solutions in terms of known functions (for example elliptic) and six that have become known as the Painlevé transcendents. Consequently, those equations that have the property that solutions have only movable poles, are said to have the Painlevé property.

This property has an intimate connection with completely integrable p.d.es. Ablowitz *et al.* (1980) were the first to show this. They were able to show that reductions of an integrable p.d.e. by, for instance, a similarity transformation, produced an o.d.e. that had the Painlevé property. The m.K.d.V. equation in fact becomes the second Painlevé transcendent when in similarity form. A more general approach would be to see if a Laurent series, as in (5.1), could be found for the p.d.e. without any reduction to an o.d.e. Weiss *et al.* (1983) have indeed shown that this can be done if one replaces the $t - t_0$ by a function $\varphi(x, t)$. This is needed because the singularities of functions of more than one complex variable cannot be localized. Hence we look for a Laurent series of the form

$$u(x, t) = \varphi^{-n} \sum_{m=0}^{\infty} u_m \varphi^m, \tag{5.3}$$

except that the coefficients u_m must now be functions of x and t. I will again use the K.d.V. equation as an example. To balance the uu_x and u_{xxx} terms we must obviously have $n = 2$. Substitution of (5.3) into the K.d.V. equation yields a recursion relation

$$\varphi_x^2 (m+1)(m-4)(m-6)\, u_m = F(u_{m-1}, \ldots u_0; \varphi_t, \varphi_x \cdots). \tag{5.4}$$

Clearly, this recursion relation is not defined when $m = 4$ and 6 and so u_4 and u_6 are arbitrary. These are called resonances. We find

$$m = 0, \quad u_0 = 2\varphi_x^2, \tag{5.4a}$$

$$m = 1, \quad u_1 = -2\varphi_{xx}, \tag{5.4b}$$

$$m = 2, \quad \varphi_x \varphi_t - 6u_2 \varphi_x^2 + 4\varphi_x \varphi_{xxx} - 3\varphi_{xx}^2 = 0, \tag{5.4c}$$

$$m = 3, \quad \varphi_{xt} - 6u_2 \varphi_{xx} + 6u_3 \varphi_x^2 + \varphi_{xxxx} = 0. \tag{5.4d}$$

If the resonance functions are set to zero: $u_4 = u_6 = 0$, and we demand that u_2 is *another* solution of the K.d.V. equation, then we find that all $u_m = 0$ for $m \geqslant 3$ and higher compatibility conditions are satisfied. Putting this all together we find

$$u = u_2 - 2\frac{\partial^2}{\partial x^2}\ln\varphi, \tag{5.5a}$$

$$\varphi_x\varphi_t - 6u_2\varphi_x^2 + 4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2 = 0, \tag{5.5b}$$

$$\varphi_{xt} - 6u_2\varphi_{xx} + \varphi_{xxxx} = 0. \tag{5.5c}$$

The Laurent series has truncated itself and φ and two solutions of the K.d.V. (u and u_2) are coupled together in three equations. This has a close relation with the Schrödinger spectral problem. Let us eliminate u_2 from (5.5 b) and (5.5 c), and find an expression for φ that we can integrate once and introduce a constant 6λ. Then we eliminate t-derivatives in the same equations and use the same result. We find

$$\left(3\frac{\partial^3}{\partial x^3} - 6\frac{\partial}{\partial x}u_2 - 6u_2\frac{\partial}{\partial x}\right)\varphi_x = -12\lambda\phi_{xx}. \tag{5.6}$$

This equation is none other than a squared eigenfunction relation for the Schrödinger problem! Let $\varphi_x = \psi^2$, then ψ satisfies

$$-\partial^2\psi/\partial x^2 + u_2\psi = \lambda\psi. \tag{5.7}$$

We conclude that this more general Painlevé method does indeed give the correct spectral problem for a completely integrable p.d.e. While not identical to the o.d.e. method in the sense that its Laurent series self-truncates automatically, the idea is nevertheless a generalization of the o.d.e. method and therefore generalizes Kovalevskaya's original idea. These results also fit in with Hirota's direct method (Hirota 1980). Equations (5.5) can be thought of as constituting a Bäcklund transformation. Let u and u_2 be two adjacent solutions in a set $\{u^{(i)}\}$ of solutions of the K.d.V. equation. Let

$$f^{(i)} = \varphi_{i-1}f^{(i-1)}, \tag{5.8a}$$

then the $f^{(i)}$ satisfy (Gibbon *et al.* 1985)

$$[(\mathrm{D}_x^4 + \mathrm{D}_x\mathrm{D}_t)f^{(i)}(x,t)f^{(i)}(x',t')]_{\substack{x'=x\\t'=t}} = 0 \tag{5.8b}$$

for every i, where $u^{(i)} = -2(\ln f^{(i)})_{xx}$ and

$$\mathrm{D}_x = \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \quad \text{and} \quad \mathrm{D}_t = \frac{\partial}{\partial t} - \frac{\partial}{\partial t'}. \tag{5.8c}$$

Equation (5.8 b) is the Hirota (1980) reduction of the K.d.V. equation to bilinear form and furthermore equation (5.5 a) is the Crum transformation (Crum 1955), which is used to add in or take out an eigenvalue of the Schrödinger operator.

The idea of the Painlevé property is used in Dr Ward's paper in this symposium. While solutions of the self-dual Yang–Mills equations can be shown to be meromosphic in any gauge (Ward 1984), difficulties arise in proving this result for the K.d.V. equation because it is hypothetically possible that other solutions might exist that do not have the simple Laurent type expansion (5.3). While these ideas are by no means entirely understood, the fact that integrable systems, in some loose sense, possess the Painlevé property is an interesting fact, particularly since the procedure (5.3)–(5.8) yields a squared eigenfunction relation for the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Schrödinger spectral problem in a natural way and is also intimately related to Hirota's method. While the Painlevé test can be useful, it has one drawback that nobody has yet discovered how to avoid. For instance, if we take the K.d.V. equation and change it into the variable ρ through the transformation $u = \ln \rho$, we would find that the p.d.e. in ρ would fail the Painlevé test since it would contain an essential singularity. It might be possible to remove essential singularities by some cunning transformation, but unless we know how to do so, all we can say is that the failure of the Painlevé test means that the equation is not of Painlevé type *in those variables*. There might be some transformation that transforms it into Painlevé form, but unless this can be found by luck or sleight of hand, no information is gained. Furthermore, this method gives no guarantee that all possible solutions of the equation concerned can be found from the initial Laurent series.

6. Conclusion

The only reasonable conclusion that can be made to an introductory paper such as this is to point the reader towards some of the other papers presented in this symposium and give references to some of those topics not mentioned.

While I have endeavoured to show that the highly special nature of integrable systems is no barrier to their usefulness since they can be canonical under favourable circumstances, nevertheless, an understanding of how integrable systems are affected by perturbations is important. As Ablowitz *et al.* (1974) have pointed out, the inverse scattering transform is like a form of nonlinear Fourier analysis. The spectrum, particularly the discrete part, can be likened to the 'normal modes' of the system. Perturbing the equations by adding extra terms perturbs these eigenvalues. This approach has been pursued by Kaup & Newell (1978), Keener & McLaughlin (1979) and Karpman & Maslov (1977*a–c*) with moderate success. A totally different type of problem that is, as yet, unsolved is the initial boundary value problem for these equations. Professor Keller's paper in this symposium provides an excellent example of this when considering the generation of trains of solitons in a towing tank. As a general problem, we need to solve a problem with zero initial data $u(x,0) = 0$, but with $u(x,t) = f(x)$ over some range of x for all t. This type of problem has received relatively little attention over the years and will no doubt become increasingly important.

In the papers by both Professor Frenkel and Dr Wilson, there is mention of the so-called τ-function, a name invented by Sato and co-workers at Kyoto University. The τ-function turns out to be Hirota's f-function (see equation (5.8)). Hirota (1980) used the transformation

$$u = -2\frac{\partial^2}{\partial x^2}\ln \tau, \quad \tau \equiv f, \tag{6.1}$$

to reduce the K.d.V. equation to the bilinear form

$$(\mathrm{D}_x^4 + \mathrm{D}_x\,\mathrm{D}_t)\,\tau(x,t)\,\tau(x',t')\,\Big|_{\substack{x'=x\\ t'=t}} = 0. \tag{6.2}$$

The Japanese workers have shown that this type of bilinear form arises as a consequence of the properties of the vertex operators associated with infinite-dimensional Lie algebras (Jimbo & Miwa 1983). While the straight soliton solutions are easy to find from (6.2), a large range of other solutions exist, such as polynomials in x and t for τ, which give rational solutions for u. Jacobi θ-function solutions for τ also exist among other more complicated elliptic function solutions. Dr Wilson connects some of the Japanese work on the τ-function and Kac–Moody algebras with Russian work on algebraic curves (Krichever 1977) from which some of these θ-function solutions originally arose. Dr Wilson and Dr Ercolani both mention the fact that the elements of these sorts of ideas for o.d.es go back as far as Baker (1895) and Burchnall & Chaundy (1922). This only reinforces the observation that the modern subject of integrable systems has indeed a long pedigree and is, to a certain extent, rediscovering and generalizing some old results from the last quarter of the last century. What were once thought of as old and fossilized results are now turning out to be the precursors of new and exciting work.

Appendix I

The most rigorous derivation and consideration of inverse problems in this context is by Deift & Trubowitz (1979), but the book by Ablowitz & Segur (1981) gives a perfectly adequate account of the ideas without the results being obscured by rigour. Here I give a deliberately crude sketch of how the Gel'fand–Levitan equation can be derived from the integral representations for Ψ and $\bar{\Psi}$ given in (2.7). As they are written, they are assumptions that we will not prove here, although the result is perfectly correct. It is also possible to establish that φ exp $(\mathrm{i}kx)$ and ψ exp $(-\mathrm{i}kx)$ and a are analytic in the upper half-plane and $\bar{\psi}$ exp $(\mathrm{i}kx)$ is analytic in the lower half-plane. Using the integral representations in (2.3), we find that

$$\Phi/a = \mathrm{e}^{-\mathrm{i}kx}+\int_x^\infty \bar{K}(x,s)\,\mathrm{e}^{-\mathrm{i}ks}\,\mathrm{d}s+\frac{b}{a}\left(\mathrm{e}^{\mathrm{i}kx}+\int_x^\infty K\,\mathrm{e}^{\mathrm{i}ks}\,\mathrm{d}s\right) \tag{I 1}$$

and so for $y > x$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi}{a}\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}k = \bar{K}(x,y)+B_{\mathrm{c}}(x+y)+\frac{1}{2\pi}\int \exp\left[\mathrm{i}k(y-x)\right]\mathrm{d}k, \tag{I 2a}$$

where
$$B_{\mathrm{c}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b}{a}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{I 2b}$$

On the assumption that $a(k)$ has simple zeros at $k = \mathrm{i}\kappa_n$ we note that the left side of (I 2a) becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi}{a}\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}k = \mathrm{i}\sum_{n=1}^{N}\frac{\varphi_n}{a_n'}\,\mathrm{e}^{-\kappa_n y}-\frac{1}{2\pi}\int_{C_R}\exp\left[\mathrm{i}k(y-x)\right]\mathrm{d}k, \tag{I 3}$$

which is obtained by a contour integration and C_R is a semicircular contour in the upper half-plane. For $y > x$, this contour integral is zero, and at the discrete eigenvalues $k = \mathrm{i}\kappa_n$ we have $a(\mathrm{i}\kappa_n) = 0$; a_n' is the residue of a from Cauchy's theorem. Since $a(\mathrm{i}\kappa_n) = 0$ we have from (2.3) that φ_n/a_n' is proportional to Ψ_n and so, replacing Ψ_n from (2.7a), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi}{a}\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}y = \mathrm{i}\sum_{n=1}^{N}c_n\exp\left[-\kappa_n(x+y)\right]+\int_x^\infty K(x,s)\sum_{n=1}^{N}c_n\exp\left[-\kappa_n(y+s)\right]\mathrm{d}s. \tag{I 4}$$

Defining
$$B(x) = \sum_{n=1}^{N}c_n\,\mathrm{e}^{-\kappa_n}+B_{\mathrm{c}}(x) \tag{I 5}$$

and using (I 4) in (I $2a$) we have finally

$$K(x,y)+B(x+y)+\int_x^\infty K(x,s)\,B(s+y)\,\mathrm{d}s=0, \tag{I 6}$$

where $\bar{K}=K$ since $\bar{\psi}(x,k)=\psi(x,-k)$. Equation (I 6) is the Gel'fand–Levitan equation ($2.8a$). To find the relation between $u(x,t)$ and $K(x,y)$ we substitute the integral representation for Ψ into (2.10) and find

$$u(x,t)=-2\,\mathrm{d}K(x,x)/\mathrm{d}x, \tag{I 7a}$$

$$K_{xx}-K_{yy}=uK. \tag{I 7b}$$

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